Indian Statistical Institute
Back Paper Examination
Analysis of several variables - MMath I

Max. Marks : 100 Time : 3 hours

State clearly the results that you use. Justify your answers. All questions carry equal marks.

(1) When do you say a function $f : U \longrightarrow \mathbb{R}^m$ defined on an open subset U of $\mathbb{R}^n$ is differentiable at $x \in U$. If f is differentiable at $x \in U$, show that the partial derivatives $\frac{\partial f_i}{\partial x_j}(x)$, $1 \leq i \leq m$, $1 \leq j \leq n$, exist and

$$df_x(e_j) = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j}(x) u_i$$

where $e_1, \dots, e_n$ and $u_1, \dots, u_m$ denote the standard ordered bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and f_i, $1 \leq i \leq m$ are the coordinate functions of f.

(2) Show that the function

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous on $\mathbb{R}^2$ and that all directional derivatives exist at $(0,0)$. Is f differentiable at $(0,0)$?

(3) What are *critical points*? Construct a differentiable function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with

$$C = \{(x, x^2) \, : \, x \in \mathbb{R}\}$$

as its set of critical points.

(4) If f is a real function defined on a convex open set $E \subseteq \mathbb{R}^n$ such that $D_1 f(x) = 0$ for every $x \in E$, prove that $f(x)$ only depends on $x_2, x_3, \dots, x_n$.

(5) What is an affine map? Let H be the parallelogram in $\mathbb{R}^2$ whose vertices are $(1,1), (3,2), (4,5), (2,4)$. Describe the affine map T that sends $(0,0)$ to $(1,1)$; $(1,0)$ to $(3,2)$; $(0,1)$ to $(2,4)$. Compute the Jacobian of T.

(6) State Green's theorem. Use Green's theorem to evaluate the integral

$$\int_\gamma (1 + xy^2)\, dx - x^2 y \, dy$$

where γ is the part of the parabola $y = x^2$ from $(-1,1)$ to $(1,1)$.